THE EVOLUTION OF . . .

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Symmetry

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The word "symmetry" is used in mathematics quite differently from in ordinary speech. In everyday life one applies it mainly to two-sided, right-left symmetry; but not so in mathematics. Admittedly, the word sometimes has a more general meaning in everyday speech. For example, everyone recognises that Figure 1 is highly symmetric, although it has no two-sided symmetry. However, this is really an exception. (The example of Figure 1 calls for some comment: in preparing this lecture it struck me that one can easily run into political or religious symbols when seeking examples of highly symmetric figures. This shows that symmetry has always had a powerful effect on people.)

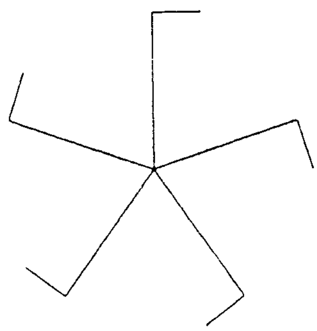


Figure 1

A second difference between symmetry in mathematics and symmetry in ordinary speech lies in the fact that *perfect* symmetry exists only in mathematics and not in real life. Here I need only allude to the fact emphasised by Hermann Weyl in his book *Symmetry*, that in western art the artist avoids the possibility of perfect symmetry and always breaks it slightly. There are beautiful examples of this, such as the famous Etruscan riders on the triclinic tomb in Corneto (Figure 2).

The picture is almost symmetric, but not quite. Perfect symmetry in art is often a little boring! In mathematics it is not so (even though in recent times mathematicians have also been interested in "near symmetries"). However, the assertion that complete symmetry never appears in reality is more fundamental than that. Look at Figure 3, for example.

Symmetries appear at first glance—and we shall come back to them—but they vanish when one looks more closely. This is immediately clear when one observes the symbols attached to the vertices—the 30 vertices have 30 different names—but even when one overlooks this, one easily finds small irregularities in the drawing that destroy all visible symmetries.

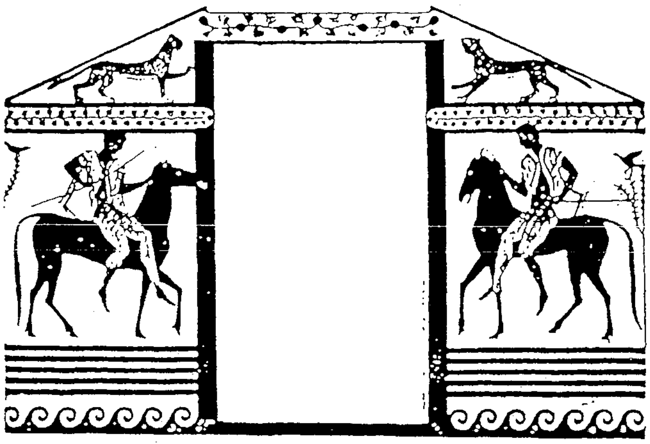


Figure 2

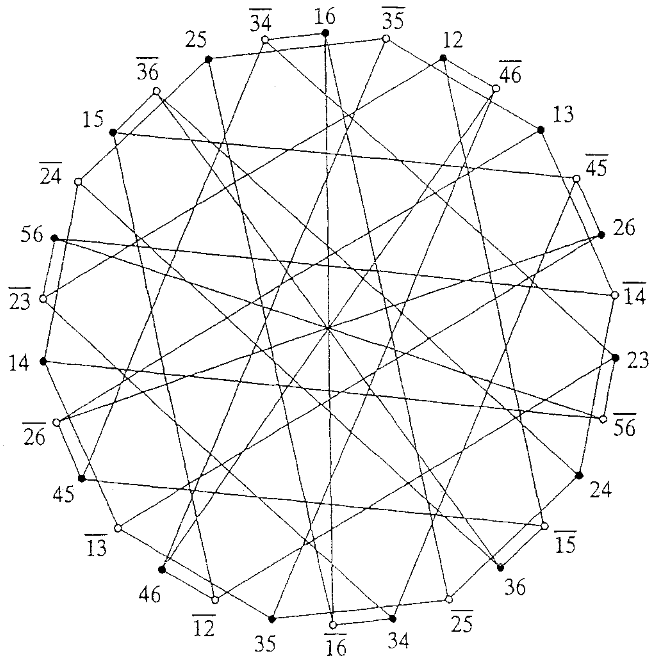


Figure 3

If one disregards the symbols and the small irregularities, however, then one immediately recognises a rotation symmetry of order 5, indeed a symmetry of order 10 if one makes no distinction between the "white" and "black" points. But much more symmetry is hidden in this figure, and it becomes apparent when one regards the figure simply as a *graph*. This means that one notices only the 30 points (lying on the boundary) and pairs of these that are connected. One may think of the 30 points, for example, as 30 people, with connections between acquaintances. The lengths and angles of the connecting lines (called edges) play no role. The hidden symmetries may then be observed as follows.

Each white point is connected to three black points, and the pairs of symbols attached to the latter three points form a so-called *partition* of the set $\{1, 2, 3, 4, 5, 6\}$. For example, the point $\overline{12}$ is connected to 46, 15, and 23, so we obtain the partition

(46)(15)(23) of the set $\{1, 2, 3, 4, 5, 6\}$. Conversely, if one takes an arbitrary partition, say (12)(34)(56), one finds that the corresponding three black points 12, 34, and 56 are connected to the same white point, in this case $\overline{23}$. Thus the white points are associated with the 15 partitions of $\{1, 2, 3, 4, 5, 6\}$ into pairs.

Now each permutation σ of the set $\{1, 2, 3, 4, 5, 6\}$ yields a permutation of the black points (since these correspond to pairs of elements of that set), as well as a permutation of the white points (regarded as partitions), which together represent a symmetry of the whole figure (regarded as a graph). For example, the permutation

$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1 \tag{1}$$

yields the symmetry

$$\begin{aligned}
&12 \to 23 \to 34 \to 45 \to 56 \to 16 \to 12,\\
&13 \to 24 \to 35 \to 46 \to 15 \to 26 \to 13,\\
&14 \to 25 \to 36 \to 14,\\
&\overline{12} = (15)(23)(46)\\
&\quad \to (26)(34)(15) = \overline{45}\\
&\quad \to (13)(45)(26) = \overline{26}\\
&\quad \to (24)(56)(13) = \overline{14}\\
&\quad \to (35)(16)(24) = \overline{25}\\
&\quad \to (46)(12)(35) = \overline{46}\\
&\quad \to (15)(23)(46) = \overline{12},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\overline{13} \to \overline{35} \to \overline{36} \to \overline{13}\\
&\overline{15} \to \overline{56} \to \overline{16} \to \overline{15}\\
&\overline{23} \leftrightarrow \overline{34}\\
&\overline{24} \to \overline{24}.
\end{aligned}$$

We remark that the resulting permutation of the white points may be simply described as being induced by the permutation

$$\bar{\sigma}: \; \bar{1} \to \bar{5} \to \bar{6} \to \bar{1}, \quad \bar{2} \leftrightarrow \bar{4}, \quad \bar{3} \to \bar{3}. \tag{2}$$

Things are similar for any permutation σ of $\{1, 2, 3, 4, 5, 6\}$, so each permutation σ becomes associated with a permutation $\bar{\sigma}$ of $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$, which usually seems very different from σ. (One notices, for example, that the σ in (1) permutes the six symbols 1, 2, 3, 4, 5, 6 cyclically, whereas $\bar{\sigma}$ in (2) has a "fixed point", namely $\bar{3}$.)

The 720 permutations of 1, 2, 3, 4, 5, 6 (and also those of $\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}$ form a *group*, the so-called *symmetric group* S_6. The correspondence $\sigma \leftrightarrow \bar{\sigma}$ is a symmetry —or, as one says in mathematical language, an *automorphism*—of the group S_6. The existence of this so-called outer automorphism of S_6 is a well-known and remarkable phenomenon, which has no analogue when 6 is replaced by another integer. By means of the above method (inducing symmetries by permutations) we have obtained 720 symmetries of the graph in Figure 3; there are 1440 when one also combines them with the central symmetry $12 \leftrightarrow \overline{12}, 13 \leftrightarrow \overline{13}, \ldots, 56 \leftrightarrow \overline{56}$, which indeed is a symmetry only when white and black points are not distinguished.

One sees that the symmetry properties of a figure depend very much on how one visualises the figure: in Figure 3, whether one pays attention to irregularities in drawing, lengths of lines, the difference between white and black points etc. This leads in a natural way to the concept of a *mathematical object*, namely a thing for which the properties one intends to consider are prescribed at the beginning. Such objects can have proper symmetries. In daily life, on the other hand, it is usual and often necessary to consider all aspects of a thing, as far as possible, and this naturally destroys all symmetry.

In mathematics and physics, remarkable symmetries are often hidden. Figure 3 showed us two examples of this: on the one hand there are 720 symmetries of the graph, which come from permutations of $\{1, 2, 3, 4, 5, 6\}$ and are not at all apparent, apart from the 72 degree rotation and its multiples; on the other hand there is the outer automorphism of S_6. One of the most interesting tasks of the mathematician is to discover such hidden symmetries. We shall give further examples.

In order to introduce the next example, we present two apparently elementary problems. The first is: in how many ways may a natural number N be decomposed into sums of odd numbers? One must frank a letter with N cents, say, using stamps of denominations 1 cent, 3 cents, 5 cents, etc., and we ask in how many ways this is possible. With $N = 6$, for example, there are four solutions. The number M of solutions grows with N according to a law that is not at first easy to ascertain (Figure 4).

The second problem formulated in Figure 4 is not so easy to explain in terms of stamps, but let us try. The stamps are of two kinds: "normal" stamps whose values are even numbers, and "special" stamps whose values are the so-called triangular numbers $\{1, 3, 6, 10, \dots\}$, and they are subject to the condition that at most one special stamp is used. Again we ask for the number of combinations of such stamps that add up to N cents.

It is remarkable that these two very different problems have the same answer: for every N the numbers of combinations of the two types are equal (Figure 4). This corresponds to a well-known and deep formula of Gauss (Figure 5).

PROBLEM 1. In how many ways may a given positive integer N be expressed as a monotonically decreasing series of positive odd integers?

Example.

$$\begin{aligned} 6 &= 1 + 1 + 1 + 1 + 1 + 1 \\ &= 3 + 1 + 1 + 1 \\ &= 3 + 3 \\ &= 5 + 1 \end{aligned}$$

PROBLEM 2. In how many ways may the number N be expressed as an ordered sum whose first term is of the form $n(n + 1)/2$, while the other terms form a monotonic increasing sequence of positive even integers?

Example.

$$\begin{aligned} 6 &= 0 + 2 + 2 + 2 \\ &= 0 + 2 + 4 \\ &= 0 + 6 \\ &= 6 \end{aligned}$$

Both problems have the same answer M:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	...
M	1	1	2	2	3	4	5	6	8	10	12	14	17...	

Figure 4

The equivalence of Problems 1 and 2 corresponds to the formula of Gauss

$$\frac{\eta(q^2)^2}{\eta(q)} = \sum_{n=-\infty}^{+\infty} q^{2(n+\frac{1}{4})^2}$$

where

$$\eta(q) = q^{1/24} \cdot \prod_{n=1}^{\infty} (1 - q^n)$$

Figure 5

Should one now say that the whole situation is completely understood as soon as a proof of the Gauss formula is produced? I do not think so. One gains a deeper insight by constructing a mathematical object that reflects the two problems, and in such a way that their equivalence (that is, the equality of their number of solutions) corresponds to a symmetry (no doubt hidden) of the object. The existence of this symmetry then explains not only the equivalence of the problems but also the formula of Gauss. Such an object was found by Frenkel, Kac, Lepowsky, et al. in the representation theory of certain Kac-Moody-Lie algebras.

To introduce our last example of a hidden symmetry, consider the lattices shown in Figures 6 and 7, whose symmetries we investigate briefly. Both have the so-called translation symmetries, which we wish to disregard here. To exclude them we could, for example, fix a point of the lattice. The symmetry group then becomes finite, and indeed of order 12 for Figure 6—one finds 6 rotations and 6 reflections that leave the lattice invariant—and order 8 for the lattice in Figure 7 (which is therefore somewhat less symmetric than the first).

Symmetry groups of lattices have excited great interest in recent times, for number theory reasons among others. If one investigates lattices in three-dimensional space, four-dimensional space, etc. from the standpoint of symmetry, there suddenly appears in twenty-four-dimensional space a quite special lattice, which is highly symmetric. It is the so-called Leech lattice (unknown until 30 years ago, which seems scarcely conceivable to many mathematicians today). When J. Leech discovered that lattice he did not know that it had extraordinary symmetry properties. He was interested in quite a different problem, namely, dense packings of spheres. Looking at the lattice as he did, with the construction he gave, the lattice does not appear to be particularly symmetrical. The construction may be

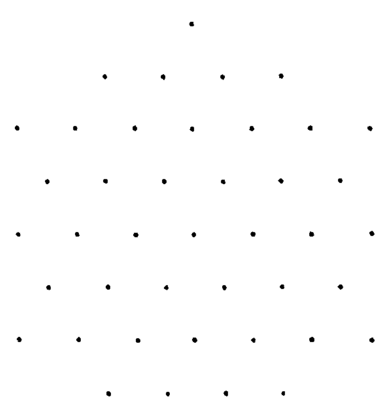


Figure 6

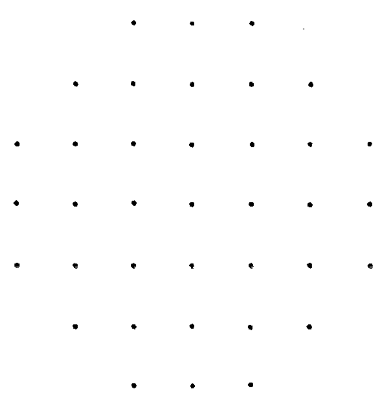


Figure 7

roughly described as follows: one begins with a nice symmetric lattice, the rectangular one, removes some points, and replaces them with others. Both modifications are asymmetric and would appear to partially destroy the original symmetry. However, the process produces new symmetries that are not immediately visible: they are "hidden symmetries". J. H. Conway was the first to notice that the Leech lattice has an enormous symmetry group, a group of order 8 315 553 513 086 720 000. I know of no explicit construction of the Leech lattice that allows the full symmetry to be immediately seen: there always remain hidden symmetries that are difficult to find.

A last, famous, example is the following. It is known that in the space of one hundred and ninety six thousand eight hundred and eighty three dimensions there is a wonderful lattice whose symmetry group has order

$$808\ 017\ 424\ 794\ 512\ 875\ 886\ 459\ 904\ 961\ 710\ 757\ 005\ 754\ 368\ 000\ 000\ 000$$
$$= 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

Of course, no one has ever really seen this lattice: we know that it exists, but an explicit construction is lacking. Nevertheless, one can construct the symmetry group of the lattice, the so-called *monster group* M of R. Griess and B. Fischer. Here again, finding a hidden symmetry is an essential step in the construction. The group M has a certain subgroup of order

$$2^{46} \cdot 3^{9} \cdot 5^{4} \cdot 7^{2} \cdot 11 \cdot 13 \cdot 23,$$

which is well understood. To generate M, Griess constructs in the space of 196 883 dimensions a certain object, a so-called algebra, which has this smaller group as (part of) its symmetry group. Then with greater difficulty he determines another symmetry, which is truly hidden. Together with the known subgroup, the hidden symmetry generates the group M. The author showed later that M is the *full* symmetry group of the Griess algebra.

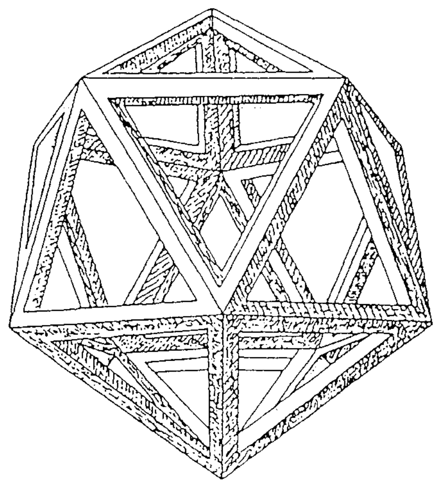
Now it is natural to ask: why is one particularly interested in this monster group? Is it more than a beautiful game? I would like to show that the answer is decidedly positive.

Since Galois, the question of finding all possible symmetry types, and hence all existing groups (and here I always mean *finite* groups) has had a clear meaning. It is indeed a natural question, even a fundamental one, but it turns out not to be a reasonable problem, as I now briefly explain.

It is well known that each natural number is a product of *prime numbers*, which are therefore the "atoms" of number theory. Finite group theory also has its "atoms", which are called *simple groups*: each group is composed from such atoms in a certain way. But while the composition process in number theory is nothing else but the ordinary multiplication, in group theory the corresponding process—called "extension"—is considerably more complicated and diverse: here, given atoms (simple groups) may be combined in many ways, and when the number of constituents and their mutual "reactivities" is large, the totality of combinations becomes completely beyond apprehension.

Still, there remains the obviously natural question of enumerating at least all the finite simple groups. Until 40 years ago even this problem was considered unrealistic, yet it has recently been solved completely. The proof has not been completely written down yet, despite the production of thousands of pages (the combined work of many specialists coordinated by D. Gorenstein), but the result is astounding. As may be expected, there are infinitely many finite simple groups, but they can all be described in a concise and unified way apart from 26 exceptions, which lie outside this nice framework and are called *sporadic groups*. Here the monster group plays a special role: it is the largest sporadic simple group and was the last discovered. In addition, it has remarkable and mysterious number-theoretic properties. Understanding these properties is one of the most fascinating problems in finite group theory today. It is worth mentioning that, in a recent work of I. Frenkel, J. Lepowsky, and A. Meurman on this theme, the hidden symmetry between the two problems in Figure 4 plays an essential role.

To conclude, I show another beautiful and well-known figure, an *icosahedron*. The psychological effect of highly symmetric objects, mentioned at the beginning, is reflected in the increasingly frequent appearance of the icosahedron in our publicity-oriented world. However, the model shown here (Figure 8) is particularly worthy of respect, since it is is supposedly due to Leonardo da Vinci.



Icoſaedron Epipedon Cenon

Icoſaedron Planum Vacuum

Figure 8

It is an easy exercise to determine all the symmetries of the icosahedron: one finds 60 rotations and the 60 products of these with the central symmetry. Now imagine the space of 196 883 dimensions, containing a crystal somewhat like the icosahedron, except that it has

808 017 424 794 512 875 886 459 904 961 710 005 754 368 *billion*

rotational symmetries. These symmetries make up the monster group, which the reader can now begin to imagine—at the same time, perhaps, gaining some impression of the beauty of symmetries in mathematics.

ACKNOWLEDGMENTS. This is an adaptation of the text of a lecture given on 28 November 1986 in Bonn (see *Mathematische Betrachtungen*, Bouvier Verlag, Bonn, 1988, pp. 32–44. I heartily thank Bouvier Verlag for permission to reproduce part of this text). The main alterations are the replacement of Figures 3 and 4 by a new Figure 3, essentially richer in symmetry properties, which I particularly dedicate to Herr Dr. H. Götze.

TRANSLATOR'S NOTE. The adaptation was published in *Miscellanea Mathematica*, edited by P. Hilton, F. Hirzebruch, and R. Remmert, Springer-Verlag, Heidelberg, 1991; this translation is published with the permission of the copyright holder, Springer-Verlag. Following an approach from the MONTHLY, Professor Tits has kindly agreed to allow this translation to be published.

REFERENCES

1. Robert L. Griess, Jr., *Twelve Sporadic Groups*, Springer-Verlag, New York, 1998.
2. Thomas M. Thompson, *From Error-correcting Codes through Sphere Packings to Simple Groups*, Mathematical Association of America, Washington, DC, 1983.

A Reality Check for Mathematicians

You need to take a break from mathematics if:

- You hear the name "Simpson" and immediately think of parabolic approximations of the integral.
- You think that "prime time TV" refers to what's on at 2, 3, 5, 7, and 11 o'clock.
- You thought the movie Matrix had something to do with linear algebra.
- In the movie Casablanca you thought Humphrey Bogart was saying "Here's looking at Euclid".
- When someone asks "What's your sign?", you wonder if you're a positive or a negative.

Submitted by David Sprows, Villanova University