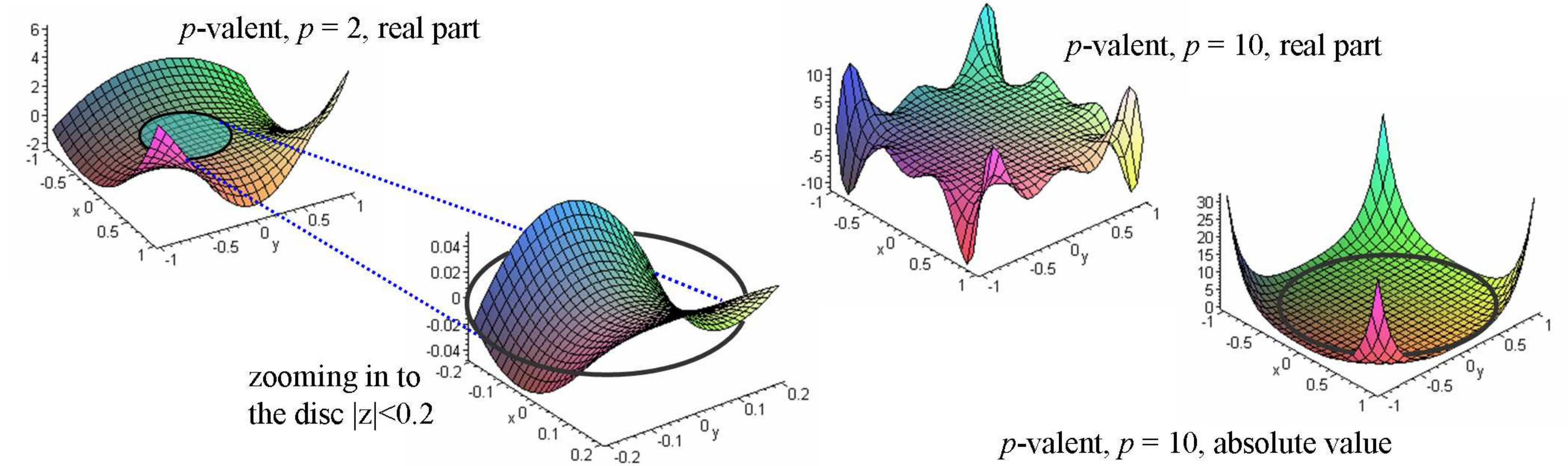
Cartwright's Theorem *For every integer $p \geq 1$, there is a constant C_p such that any function $f(z)$, analytic and p-valent in the disc $|z| < 1$, and expressed in series form as $f(z) = \sum_{i=0}^{\infty} a_n z^n$, is bounded as*

$$|f(z)| \leq \frac{\max_{0 \leq i \leq p} |a_i|}{(1-r)^{2p}} C_p,$$

in absolute value, for any $r < 1$, for all z in the disc $|z| \leq r$.



For an integer $p \geq 1$, a function is said to be p-valent in a region U if it takes no value more than p times on U and takes some value exactly p times (otherwise it would be of lower valency). A class of interest is $A(p)$, $p \geq 2$, of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$, which are p-valent and analytic (everywhere complex differentiable) on the disc $|z| < 1$. On the left, the 2-valent function $f(z) = z^2 + \sum_{k=3}^{\infty} 2^k z^k / k!$ is shown. Its real part (the imaginary part is similar but 'inverted') appears to be a flat surface except near the circumference of the unit disc, but the magnification, bottom left, shows this is an illusion. Bottom right, the absolute value of a 10-valent version is shown. It climbs very rapidly beyond the confines of the unit disc but, obeying Cartwright's Theorem, remains bounded within.

G.H. Hardy would comfort himself, when 'forced to listen to pompous and tiresome people, "Well, I have done one thing *you* could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms."' Ramanujan returned to India to die in 1919, the year that Mary Cartwright (1900–1998) arrived in Oxford as a student. She was herself to collaborate at the highest level with Littlewood over a period of many years. This famous theorem was proved in 1930 in response to a problem set by Littlewood in a function theory class.

Web link: `www.ams.org/notices/199902/mem-cartwright.pdf`

Further reading: *Multivalent Functions, 2nd Edition* by W.K. Hayman, Cambridge University Press, 2008. The Hardy quote is from A Mathematician's Apology.