An Invitation to Mathematics

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Editors

An Invitation to Mathematics

From Competitions to Research

Springer

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Preface: What is Mathematics?

Günter M. Ziegler

This book is an Invitation to Mathematics.

But *What is Mathematics*? This is a question that asks us for a definition. You could look in *Wikipedia* and find the following:

> *Mathematics is the study of quantity, structure, space, and change. Mathematicians seek out patterns, formulate new conjectures, and establish truth by rigorous deduction from appropriately chosen axioms and definitions.*

Quantity, structure, space, and change? These words outline a vast field of knowledge — and they are combined with a very narrow, mechanistic, and, frankly, quite boring description of "what mathematicians do". Should "what mathematicians do" really be a part of the definition?

The definition given by the German *Wikipedia* is interesting in a different way: it stresses that there is no definition of mathematics, or at least no commonly accepted one. I translate:

> *Mathematics is the science that developed from the investigation of figures and computing with numbers. For mathematics, there is no commonly accepted definition; today it is usually described as a science that investigates abstract structures that it created itself for their properties and patterns.*

Is this a good definition, a satisfactory answer to the question "What is Mathematics"? I believe that *Wikipedia* (in any language) does not give a satisfactory answer. At the same time, and much more importantly, high school curricula do not give a satisfactory answer. Even the famous book by Richard Courant and Herbert Robbins entitled "What is Mathematics?" (and subtitled "An Elementary Approach to Ideas and Methods") does not give a satisfactory answer.

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Perhaps it is impossible to give a good definition in a sentence or two. Indeed, I claim that there cannot be one single answer that we could be content with: mathematics in the 21-st century is a huge body of knowledge and a very diverse area of study. There are thus so many ways to experience mathematics — the arenas of national and international competitions, and research experiences that range from years spent working in solitude (think of Andrew Wiles, who proved Fermat's Last Theorem, or Grigori Perelman, who proved the Poincaré conjecture) to coffee break discussions at conferences to massive collaborations on internet platforms (such as the POLYMATH projects initiated by Michael Nielsen, Timothy Gowers, Terence Tao, and others).

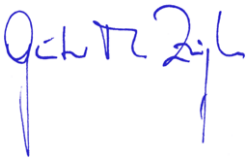
But perhaps the English *Wikipedia* is right in one aspect — that in approaching the science called mathematics one should look at the people who do mathematics. So *what is mathematics as an experience*? What does it mean to *do* mathematics?

This book is an invitation to mathematics comprised of contributions by leading mathematicians. Many of them were initiated to mathematics, and led to mathematics research, through competitions such as the mathematical olympiads — one of the ways to get attracted to and drawn into mathematics. This book builds a link between the "domesticated" mathematics taught at high schools or used in competitions and the "wild" and "free" world of mathematical research. As a former high school student, successful participant at competitions such as the IMO 1981, and now professor of mathematics who is doing research and who is active in communicating mathematics to the public, I have personally experienced all these kinds of mathematics, and I am excited about this book and the link that it provides.

The starting point of this book was an event that I had the pleasure of hosting (jointly with Martin Grötschel), namely the 50-th International Mathematical Olympiad, held in Bremen in 2009, at which several premier IMO gold medal winners got on stage to talk about the mathematics that they studied, the mathematics that they are studying, and the mathematics that they are interested in.

All this is reflected in this volume, which contains some of these IMO presentations, as well as other facets of the mathematics research experience. It was put together with admirable care, energy, and attention to detail by Dierk Schleicher (one of the chief organizers of the 50-th IMO in Bremen) and Malte Lackmann (a successful three-time IMO participant). Let me express my gratitude to both of them for this volume, which I see as a book-length exposition of an answer to the question "What is Mathematics?" — and let me wish you an informative, enjoyable, and (in the proper sense of the word) *attractive* reading experience.

Berlin, November 2010



Günter M. Ziegler

Welcome!

Dear Readers,

we are pleased that you have accepted our *Invitation to Mathematics*. This is a joint invitation by a number of leading international mathematicians, together with us, the editors. This book contains fourteen individual invitations, written by different people in different styles, but all of us, authors and editors alike, have one thing in common: we have a passion for mathematics, we enjoy being mathematicians, and we would like to share that enjoyment with you, our readers.

Whom is this book written for? Broadly speaking, this book is written for anyone with an interest in mathematics — yes, for people just like you. More specifically, we have in mind young students at high schools or universities who know mathematics through their classes and possibly through mathematics competitions that they have participated in, either on a local level or all the way up to the level of international olympiads. Mathematics has different flavors: the kind of mathematics found at high school is distinctly different from that found at competitions and olympiads, and both are quite different from mathematics at the research level. Of course, there are similarities too — after all, it's all mathematics we're talking about.

The idea of this book is to allow professional research mathematicians to share their experience and some aspects of their mathematical thinking with our readers. We made a serious effort to reach out to you and write at a level that, for the most part, should be accessible to talented and, most importantly, interested young students in their final years of high school and beyond. Quite importantly, this book is also meant to address high school math teachers, in the hope that they find our invitation interesting as well and share this interest with their students. And of course, we hope that even active research mathematicians will find this book inspiring and, in reading it, will gain new insights into areas outside of their specialization — just as we learned quite a bit of mathematics ourselves in the process of editing this book.

Fourteen invitations to mathematics. You will find that the individual invitations in this book are as varied as the personalities of their authors and their mathematical tastes and preferences: mathematics is a place for very different people. All of our fourteen invitations are independent from each other, and you are welcome to browse through them and start with those that you like best, or that you find most accessible, and continue in your preferred order — much as the white "random path" on the book cover that connects the pictures from the different contributions. (We ordered the contributions by attempting to bring together those that discuss related topics, but of course there would have been many equally good ways to order them.) If you get stuck at one particular invitation, you may want to continue with another. You may discover, though, that some of those that you found difficult at first may become easier, and perhaps more beautiful, once you've learned some more mathematics, or gotten more experience with other invitations, or simply had some time to digest them. Indeed, a number of our contributions are invitations to active reading, and ask you to think along seriously: after all, thinking is what most of us do for much of our professional research time.

Although we encourage you to start by reading those invitations that you prefer, we would like to equally strongly encourage you to fully use the opportunity that this book provides, with the broad area of mathematics that it covers. In high school or during competitions and olympiads, you may have developed preferences for some areas of mathematics or dislikes for others; but mathematics is a broad and rich field, and it is dangerous to become specialized too early, before having seen the beauty of many other mathematical areas. We have often spoken to young university students who thought they were sure they wanted to work in area X and thus refused to take courses in other areas, and advised them to get at least some background in areas Y and Z. Often enough, it turned out that these areas weren't so bad after all, and the students ended up doing their research in areas Y′ or Z′, or possibly Ω. And even for those few who, after having explored different branches of mathematics, ended up working in exactly the area X that they liked best as a young student, it can only be good to take along as many ideas from other areas of mathematics as possible. In modern mathematics, there is increasingly more interaction between different branches that seemed to be drifting apart some time ago. This can be seen quite well in the articles of our book: many contributions cover (apparently) quite different aspects of mathematical research and show surprising links between them. In addition, there are many links between the different contributions, so that you will often have the feeling of meeting similar ideas in quite different contexts. Rather than telling you where, we encourage you to read and discover this on your own.

To paraphrase the spirit of the book, the title is *not* "Fourteen invitations to mathematics", but "An invitation to mathematics", and we hope that you can get a glimpse of mathematics that has as much breadth and variety as we managed to fit between the covers of this book. (Mathematics itself is much broader, of course, and we thought of many further contributions.

If you feel that an important aspect of mathematics is missing here, and that we overlooked a particular person that should have contributed another invitation to mathematics, then please let us know — and help us convince this person to share this invitation with us for the next edition of this book!)

The inspiration for this book. This book was inspired by the 50-th International Mathematical Olympiad that took place in 2009 in Bremen, Germany. Both of us were quite involved in this olympiad: one as a senior organizer, the other as a participant.

One highlight of this olympiad was the 50-th IMO anniversary celebration ceremony, to which six of the world's leading international research mathematicians were invited, all of whom had personal experience with the IMO: Béla Bollobás, Timothy Gowers, László Lovász, Stanislav Smirnov, Terence Tao, and Jean-Christophe Yoccoz. *All six accepted our invitations!* They gave wonderful presentations and were celebrated by the IMO contestants and delegates like movie stars. We tried to provide ample opportunity for IMO contestants and delegates to get in contact with our guests of honor and to have a chance to interact personally with them. This was a most memorable and exciting event that created lasting memories for all of us. We hope that this spirit of personal interaction and invitation will also shine through in this book and its individual contributions.

In addition to the contributions of these guests of honor, three more of our invitations have their roots at the IMO 2009: over the course of three evenings, while the solutions of the contestants were being evaluated, we offered mathematics talks to them (given by Michael Stoll, Marcel Oliver, and Dierk Schleicher). Another contribution (by Alexander Razborov) is based on a lecture series given at the "Summer School on Contemporary Mathematics" held in Dubna/Moscow in 2009. Whatever their inspirations, all contributions were written specifically for this occasion (earlier versions of the contributions by Bollobás, Gowers, Lovász, Smirnov, Tao, and Yoccoz appeared in the report of the 50-th IMO).

This book goes far beyond a single event, exciting as it was, and tries to build lasting links between high schools, competitions, and mathematical research. To use a metaphor of József Pelikán, chairman of the IMO advisory board, research mathematics is like wildlife in uncharted territory, whereas olympiad problems are like animals in a zoo: even though they are presented as animals from the wild, they are constrained to a very restrictive cage. No lion can show its full splendor and strength in the few square meters enclosed by its cage, just as mathematics cannot show its full beauty within the rigid boundaries of competition rules. For young students who have been successful at olympiads, it is important that they learn to leave the olympiad microcosm, to get used to dealing with real mathematical wildlife, and to accept new challenges.

Advertising mathematics, or being a mathematician. We thought about using this introduction to advertise mathematics, including a recitation of the usual

claims about how important mathematics is and how much our culture is built upon mathematical thinking. However, we believe that our readers do not need to be convinced, and that the invitations speak for the beauty and value of mathematics by themselves. Nevertheless, we are aware that many students have parents or counselors who tell them that they should study something that will one day earn them money or that has safer job prospects. To them, we would like to say that young people will be most successful in areas that they enjoy the most, because it is only there that they can develop their full potential. Parents[1], please don't worry: all the students from various countries who wanted to become mathematicians and that we advised to pursue their goals despite the concerns of their parents have become quite successful in their fields, in academia, in industry, or in business, and none of them went unemployed.

What makes this book special. First and foremost, our authors include some of the world's leading mathematicians, who are sharing some of their mathematics with you, our readers. This book wants to build a bridge between active research mathematicians and young students; it was realized by a team of people that come from both ends of this bridge: authors, editors, and test readers.

Indeed, we have not made it easy for our authors to write their contributions: we adopted an editing style that Timothy Gowers, in the preface to his *Princeton Companion to Mathematics*, describes as "active interventionist editing". All contributions have been carefully read by us and by a team of young test readers at the age of the intended readership, and we or the authors improved whatever our team could not understand, until things became clear. In this way, we hope that contributions that were *meant to be* comprehensible to our readers actually *are*: the only way to find out was by asking a number of test readers, and that's what we did.

This resulted in numerous and substantial requests for changes in most contributions. All authors accepted these requests, and many were extremely pleased with the feedback they received from us. One author, who was initially somewhat skeptical about this process, wrote "I am extremely impressed by the quality of the job they have done — it greatly surpasses the average level of referee reports I have seen in all three major capacities (editor, author or, well, referee)". In the preface to his *Princeton Companion*, Timothy Gowers writes "given that interventionist editing of this type is rare in mathematics, I do not see how the book can fail to be unusual in a good way". With due modesty, we hope that this applies to some extent to this book as well, and

[1] Additional evidence for parents: just a year ago, the *Wall Street Journal* published a ranking of 200 jobs according to five important criteria: work environment, income, employment outlook, physical demands, and stress. The jobs investigated included such different occupations as computer programmer, motion picture editor, physicist, astronomer, and lumber jack. What are the top three jobs? In order, they are: mathematician, actuary, and statistician. All three jobs are based on a strong mathematics education. (Source: http://online.wsj.com/article/SB123119236117055127.html.)

that our readers will appreciate the outcome of the substantial efforts of our authors and our editorial team — our test readers, at least, told us many times that they did.

We would like to conclude this *Welcome* with quotations from two more of our test readers: "I never thought that the topic XY could be exciting to read; well, now I know, it can be!" Another one wrote, after reading a different contribution: "I really found this text very interesting to read; and this really means something because this is not an area I thought I was interested in!"

This is the spirit in which we would like to encourage you to read this book.

Bremen, November 2010

Malte Lackmann and Dierk Schleicher

Acknowledgements. First and foremost, we are indebted to the authors of our *Invitation to Mathematics*. Their willingness to provide their contributions and to share their personal insights, as well as their positive attitude with which they responded to our numerous requests for improvements, are greatly appreciated by us and, we hope, also by our readers. We had a number of "test readers" from the target group of students who patiently and carefully read through some or even all of the invitations, sometimes in several versions, and who helped the authors and us produce a much better book. Several of our authors specifically asked us to convey their appreciation to our test readers for their dedication and care, and we do this with great happiness and gratitude. Our most active test readers were Alexander Thomas, Bertram Arnold, and Kęstutis Česnavičius, but many more students read one or several texts and gave us valuable feedback, including Bastian Laubner, Christoph Kröner, Dima Dudko, Florian Tran, Jens Reinhold, Lisa Sauermann, Matthias Görner, Michael Meyer, Nikita Selinger, Philipp Meerkamp, and Radoslav Zlatev, as well as our colleagues and friends Marcel Oliver and Michael Stoll. We would also like to thank Jan Cannizzo, who greatly helped us take care of English language issues in the texts, and who language edited several contributions entirely; our authors specifically asked us to thank him sincerely. We are extremely grateful to Clemens Heine from Springer Verlag for his untiring and continuous factual and moral support in all kinds of circumstances; if it is ever true to say that a book would not have come into existence without the continuous support of the publishing editor, it is the case here. It has also been a pleasure to work with Frank Holzwarth from Springer Verlag who solved all our LaTeX issues in an instant.

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D.S. would like to thank his students and colleagues for their understanding when he was sometimes preoccupied while editing this book. And of course thank you, Anke and Diego, for your support and understanding all along, and for being with me.